SOME REMARKS ON IDEALS IN FUNCTION ALGEBRAS

BY

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ABSTRACT

Some applications are given of the abstract F. and M. Riesz theorem to ideals.

1. Applications of measures orthogonal to function algebras involving the abstract F. and M. Riesz theorem frequently require special hypotheses concerning those measures which are completely singular [5]. The present note is devoted to one set of problems—those concerning closed ideals—in which completely singular elements play no role. (We shall assume the reader is familiar with [4, 5], and most of our notation will be that of [5].)

Let I be a closed ideal in the closed subalgebra A of $C(X)$, X compact, and suppose A contains the constants. Trivially $\mathbb{C}+I$ is a closed subalgebra of $C(X)$, and the spectrum $\mathcal{M}_{C+I}$ of $\mathbb{C}+I$ consists of $\mathcal{M}_A$ with the hull of I identified to a point. In all that follows we shall let ϕ_0 be that point, the element of $\mathcal{M}_{C+I}$ with kernel I, and $M_{\phi_0} = M_{\phi_0}(\mathbb{C}+I)$ will represent the probability measures on X representing ϕ_0 on $\mathbb{C}+I$ (a set which includes $M_\phi(A)$ for each ϕ in hull I). Finally $P_\phi(A)$ will denote the Gleason part containing ϕ (for the algebra A); since ϕ defines an element of $\mathcal{M}_{C+I}$ as well, $P_\phi(\mathbb{C}+I)$ may well differ from $P_\phi(A)$ (see example 1.8.).

Our fundamental observation is the simple

LEMMA 1.1. *Suppose $\mu \perp I$ and μ is $M_{\phi_0}(\mathbb{C}+I)$-singular. Then $\mu \perp A$.*

The proof is a simple application of the abstract Forelli lemma [5, 1.2]. By regularity μ is carried by a σ-compact M_{ϕ_0}-null set, so that Forelli's result shows there is a sequence $\{f_n\}$ in the unit ball of $\mathbb{C}+I$ with $f_n \to 1$ a.e. $|\mu|$, while $f_n \to 0$ a.e. λ,

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all $\lambda \in M_{\phi_0}$. Since $\lambda(f_n) = \phi_0(f_n) \to 0$, we can replace f_n by $g_n = f_n - \phi_0(f_n)$ and obtain a bounded sequence in $\mathbb{C} + I$—in fact in I since $\phi_0(g_n) = 0$—for which $g_n \to 1$ a.e. $|\mu|$. But by dominated convergence $\| g_n\mu - \mu \| \to 0$ while $g_n\mu \perp A$ since $g_nA \subset I$; thus $\mu \perp A$ as asserted.(2)

REMARK 1.2. More generally the argument shows that for any M_{ϕ_0}-singular measure μ there is a bounded sequence $\{g_n\}$ in I with $\| g_n\mu - \mu \| \to 0$.

COROLLARY 1.3. *Suppose* $\phi \in \mathcal{M}_A \backslash \mathrm{hull}\, I$ $(= \mathcal{M}_{C+I} \backslash \{\phi_0\})$ *is not in the Gleason part* $P_{\phi_0}(\mathbb{C} + I)$. *Then* $M_\phi(C + I) = M_\phi(A)$, *and for* λ *therein* $H^2(\mathbb{C} + I, \lambda) = H^2(A, \lambda)$.

The H^2 assertion follows from the first by [4, 1.1]. Of course $M_\phi(A) \subset M_\phi(\mathbb{C}+I)$, so suppose $\lambda \in M_\phi(\mathbb{C} + I)$. Since $\phi \notin P_{\phi_0}(\mathbb{C} + I)$, λ is M_{ϕ_0}-singular and by our Remark there is a sequence $\{g_n\}$ in I with $\| g_n\lambda - \lambda \| \to 0$. But $\phi(g_n) = \lambda(g_n) \to \lambda(1) = 1$ so that

$$\left\| \frac{g_n}{\phi(g_n)}\lambda - \lambda \right\| \to 0,$$

while $g_n\lambda/\phi(g_n)$ represents ϕ on A: $\lambda[(g_na/\phi(g_n)] = \phi[g_na/\phi(g_n)] = \phi(a)$, $a \in A$. Thus λ represents ϕ on A.

Our next application of 1.1 shows $f = u + iv \in A$ (u, v real) lies in I if both u and v are approximable by elements of $\mathrm{Re}\, I$, in fact in a rather weak sense. It is an obvious corollary of a recent result of Lumer [6] which asserts that

$$\| f + I \| \leqq 4 \sup\{ |\lambda(f)| : \lambda \in M_{\phi_0}\}, \quad f \in A.$$

However our proof yields a stronger specialization which does not follow from this(3) (1.5 below).

(2) At only one point (1.4 below) do we require A really to be an algebra rather than a closed subspace containing I with $AI \subset I$. But no generality would be gained by the weaker hypothesis since then I is an ideal in the sub-algebra B generated by A, and our results for B yield the corresponding results for A.

(3) Lumer obtains his result as a side product of an investigation of semi-inner product methods in non-commutative Banach algebras, but a purely function algebra proof has since been found by Cole. Note that 1.4 yields a general extension of the very special [5, 3.3]: *closed ideals are determined by their orthogonal probability measures.* In accord with [2, 3] (but not [4, 5]) we take $H^\infty(A,\lambda)$ to be the w^* closure of A in $L^\infty(\lambda)$.

THEOREM 1.4. *$f = u + iv \in A$ is in I if and only if $f \perp M_{\phi_0}$.*

Before proceeding to the proof we need one *observation*: the argument of Gamelin and Lumer [3, I2.6, p. 125–6] applies (with $\mathrm{Re}(\mathbb{C} + I)$ in place of $\mathrm{Re}\, H^2$) to show that when $u \perp M_{\phi_0}$ there is a real Borel function w on X for which $e^{t(u+iw)} \in H^\infty(\mathbb{C} + I, \lambda)$ for all λ in M_{ϕ_0}, $t \in R$; indeed $u \perp M_{\phi_0}$ implies as usual [2]

$$\sup\{\mathrm{Re}\,\phi_0(f)\colon \mathrm{Re} f \leqq u,\ f \in \mathbb{C} + I\} = 0$$
$$= \inf\{\mathrm{Re}\,\phi_0(f)\colon \mathrm{Re} f \geqq u,\ f \in \mathbb{C} + I\}$$

and this allows one to choose $u_n', u_n'' \in \mathrm{Re}(\mathbb{C} + I)$, $u_n' \leqq u \leqq u_n''$ as required there. (Note that the non-negative (or, depending on n, non-positive) functions $u_{n+1} - u_n$ have norms in $L^1(\lambda)$ independent of $\lambda \in M_{\phi_0}$, since $\| u_{n+1} - u_n \|_1 = \pm \int u_{n+1} - u_n \, d\lambda$.) Finally the Ahern-Sarason estimate

$$\int |e^{t(u+iw)} - 1|^2 d\lambda = \int e^{2tu} - 1 d\lambda = O(t^2), \text{ while} \int e^{t(u+iw)} - 1 d\lambda = 0$$

leads as usual to the fact that $u + iw \in H^2(\mathbb{C} + I, \lambda)$ for all $\lambda \in M_{\phi_0}$, with $\int u + iw d\lambda = 0$.

To proceed to the proof of 1.4 note that we need only show $f \in \mathbb{C} + I$, since then $f \perp M_{\phi_0}$ implies $\phi_0(f) = 0$. By the abstract F. and M. Riesz theorem each element of $(\mathbb{C} + I)^\perp$ has its M_{ϕ_0}-singular and M_{ϕ_0}-absolutely continuous components in $(\mathbb{C} + I)^\perp$, and by 1.1 the former are all $\perp f$. Thus we need only see $\mu(f) = 0$ if $\mu \in (\mathbb{C} + I)^\perp$, $\mu \ll M_{\phi_0}$.

By our observation $u + iw \in H^2(\mathbb{C} + I, \lambda)$, and $e^{t(u+iw)} \in H^\infty(\mathbb{C} + I, \lambda)$, $\lambda \in M_{\phi_0}$, $t \in R$. If λ is an extreme point of M_{ϕ_0} then a bounded real Borel function g with $g\lambda \perp I$ is necessarily constant a.e. λ: for otherwise $\lambda = \frac{1}{2}(1 - \varepsilon g)\lambda + \frac{1}{2}(1 - \varepsilon g)\lambda$ represents λ as a convex combination of distinct elements of M_{ϕ_0} for $\varepsilon > 0$ small. On the other hand $e^{t(u+iw)}\lambda \perp I \supset e^{-tf} I$, so

$$e^{t(u+iw)} e^{-tf} \lambda = e^{it(w-v)} \lambda \perp I$$

and thus

$$(e^{it(w-v)} \pm e^{-it(w-v)})\lambda \perp I.$$

For λ extreme this implies $e^{it(w-v)}$ is constant a.e. λ, and so, as is easily seen, that $w - v = c$, a.e. λ. Since $0 = \int w - v d\lambda = c$, $w = v$ a.e. λ for all extreme λ, hence for all their finite convex combinations. So $f = u + iv = u + iw \in H^2(\mathbb{C} + I, \lambda)$

for a w^* dense subset of M_{ϕ_0}, hence for all λ in M_{ϕ_0} by [4, 1.3]. Thus by [5, 2.3] there is a sequence $\{f_n\}$ in the $\|f\|$-ball of $\mathbb{C}+I$ with $f_n \to f$ a.e. λ, all $\lambda \in M_{\phi_0}$, which implies $0=\mu(f_n)\to\mu(f)$ by dominated convergence since $\mu \ll M_{\phi_0}$, completing our proof.(4)

As noted the proof yields

THEOREM 1.5. *Let $\Phi \subset \text{hull} I$, and suppose that for each λ in M_{ϕ_0} (or just in a w^* dense subset of $M^e_{\phi_0}$, its set of extreme points) there is a $\phi \in \Phi$ and a λ' in $M_\phi(A)$ with λ, λ' not mutually singular. Then any $f = u + iv \in A$ which vanishes on Φ and has $u \perp M_{\phi_0}$ is in I.*

Given $\lambda \in M^e_{\phi_0}$ and λ' as above we have $\lambda' \in M_{\phi_0}$ and $u + iw$ (as in 1.4) an element of $H^2(\mathbb{C}+I,\lambda') \subset H^2(A,\lambda')$. The latter also contains $f = u+iv$, so $v-w$ is a real valued element of $H^2(A,\lambda')$, hence 0 a.e. λ': for $\int (v-w)^2 d\lambda'$ $(\int v - w d\lambda')^2 = 0$ since $f(\phi)=0$ and $\int w d\lambda' = 0$ for λ' in M_{ϕ_0}.

As in our proof of 1.4 we know $w - v = c_\lambda$ a.e. λ, and since λ and λ' are not mutually singular(5) $c_\lambda = 0$. By hypothesis, this holds for a w^* dense set of λ in $M^e_{\phi_0}$, and since $c_\lambda = \int w - v d\lambda = -\int v d\lambda$ on $M^e_{\phi_0}$, $\lambda \to c_\lambda$ is w^* continuous, hence $\equiv 0$ on $M^e_{\phi_0}$. So $w = v$ a.e. λ, all $\lambda \in M^e_{\phi_0}$, and the final portion of the proof of 1.4 shows $f \in \mathbb{C}+I$. So $f \in I$ since f vanishes on $\Phi \subset \text{hull} I$, while Φ is necessarily non void.

An easier result of the same sort is(6)

THEOREM 1.6. *Suppose L is a closed subspace of A containing the closed ideal I. Then $f \in A$ lies in L if it lies in the $L^2(\lambda)$-closure of L for each $\lambda \in M_{\phi_0}$.*

Since $(\mathbb{C}+I)L \subset L + I \subset L$, L is a $(\mathbb{C}+I)$-module so $\mu \perp L$ has its M_{ϕ_0}-singular and absolutely continuous components orthogonal to L by [5, 3.1]. As before the former is orthogonal to f by 1.1, so we want to see $\mu(f)=0$ if $\mu \perp L$, $\mu \ll M_{\phi_0}$.

By [5, 2.2] there is a sequence $f_n \in L$ with $\sup_{M_{\phi_0}} \int \left| |f - f_n| \right|^2 d\lambda \to 0$, and by [5, Remark p. 115] there is a subsequence $\{f_{n_j}\}$ and a sequence $\{a_j\}$ in the unit ball of $\mathbb{C}+I$ with $\|a_j f_{n_j}\| \leqq \|f\|$, $a_j f_{n_j} \to f$ a.e. λ, all $\lambda \in M_{\phi_0}$. Since $a_j f_{n_j} \in L$, $0 = \mu(a_j f_{n_j}) \to \mu(f)$ again by dominated convergence and the fact that $\mu \ll M_{\phi_0}$.

(4) When X is metric 1.4 shows there is a single λ_0 in M_{ϕ_0} for which $f \in I$ iff u and v each lie in the $L^1(\lambda_0)$-closure of ReI. Indeed if λ_n is a w^* dense sequence in M_{ϕ_0} and $\lambda_0 = \Sigma 2^{-n}\lambda_n$ then since u lies in the $L^1(\lambda_n)$-closure of ReI, $\int u d\lambda_n = 0$, whence $u \perp M_{\phi_0}$.

(5) As is evident at this point we could equally well assume we have a chain $\lambda_1, \dots \lambda_n$ in $M^e_{\phi_0}$ with $(\lambda_i, \lambda_{i+1})$ and (λ, λ_1), (λ_n, λ') pairs of not mutually singular measures.

(6) Lumer's methods also yield this result [6].

If $F \subset X$ is a peak set for I, with $h \in I$ peaking on F, then $\lambda F = \lim \lambda(h^n) = 0$, $\lambda \in M_{\phi_0}$; and the same applies to an intersection of peak sets. As a final observation on ideals, really unrelated to the preceding results, we want to record

PROPOSITION 1.7. *If $F \subset X$ is an intersection of A-peak sets which carries no element of M_{ϕ_0} then F is an intersection of I-peak sets.*

Let $F = \cap F_\alpha$, with F_α a peak set for A. Then we can find a finite intersection $\cap F_{\alpha_i}$ which carries no element of M_{ϕ_0} by virtue of the w^* compactness of M_{ϕ_0}. Thus F is an intersection of peak sets of A each carrying no element of M_{ϕ_0}, and it suffices to prove 1.7 when F is an A-peak set.

So let $f \in A$ peak on F. The proof of Bishop's peak set lemma [1, p. 631; 2] shows that if we have $h \in I$ with $h|F = 1$ then there is a $g \in A$ with gh peaking on F. Now $I|F$ is an ideal in the closed subalgebra $A|F$ of $C(F)$, and if no such h exists there is a $\phi \in \mathscr{M}_{A|F}$ annihilating $I|F$. But then F carries a probability measure λ representing ϕ, which of course lies in M_{ϕ_0}.

Example 1.8. The following shows that A-parts and $\mathbb{C} + I$-parts may differ (even when $\mathscr{M}_A = \mathscr{M}_{C+I}$). A more complicated example was pointed out to me earlier by Cole, Gamelin and Garnett.

Let A be the disc algebra, $X = T^1$, and λ_0 normalized Haar measure. Let I be the closed ideal $\exp[(z+1)/(z-1)]\, M$, where M is the maximal ideal of functions vanishing at 1 (cf. [6, pp. 83–84]) so $\mathscr{M}_{C+I} = \mathscr{M}_A$. Since $I \neq M$ we have a λ in M_{ϕ_0}, $\lambda \neq \delta_1$, the unit point mass at 1, by footnote 2 say. So $j\lambda \neq 0$ for some $j \in I$, and since $j\lambda \perp A$ so that $j\lambda \ll \lambda_0$, we conclude that λ and λ_0 are not mutually singular.

Thus $0 \in P_{\phi_0}(\mathbb{C} + I)$, while $P_{\phi_0}(A) = \{1\}$; indeed $P_{\phi_0}(\mathbb{C} + 1)$ is precisely the open disc plus 1, as is easily seen. We should perhaps note that 1.5 cannot be applied here since $M_{\phi_0}(A) = \{\delta_1\}$; it would be applicable to an ideal with hull $\not\subset T^1$ however.

2. We conclude with a fact unrelated to ideals.

THEOREM 2.1. *Suppose $A \subset B$ are closed subalgebras of $C(X)$ containing the constants, $\phi \in \mathscr{M}_B$, and a w^* dense set of λ in $M_{\phi_0}(A)$ are $M_\phi(B)$-absolutely continuous. If $f = u + iv \in B$ has u constant on $M_\phi(A)$ (in particular if $u \in \operatorname{Re} A)^-$) then $f \in H^2(A,\lambda)$, all $\lambda \in M_\phi(A)$.*

For the proof we can assume $\phi(f) = 0$, and thus $u \perp M_\phi(A)$. By our observation after 1.4 there is a real Borel function w with $u + iw \in H^2(A,\lambda)$, all λ in $M_\phi(A)$, with $\int u + iw d\lambda = 0$.

Now for $\lambda \in M_\phi(B) \subset M_\phi(A)$ we have $u + iv$ and $u + iw$ in $H^2(B,\lambda)$, so $v - w$ is a real element of $H^2(B,\lambda)$ of mean 0, hence $=0$ a.e. λ: for $\int (v-w)^2 d\lambda = (\int v - w d\lambda)^2 = 0$. Thus $\{x: w(x) \neq v(x)\}$ is $M_\phi(B)$-null.

For λ in our w^* dense subset of $M_\phi(A)$ we have $f = u + iv = u + iw$ in $L^2(\lambda)$ so $f \in H^2(A,\lambda)$, and this extends to all λ in $M_\phi(A)$ by [4, 1.3].

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